THE METHOD OF GEOMETRICAL OPTICS FOR DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER AS APPLIED TO LOW-FREQUENCY PLASMA OSCILLATIONS

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The theory of oscillations of a spatially inhomogeneous plasma [1] draws substantially on the theory of geometrical optics as applied to differential equations of the second order. The theory of asymptotic solutions for equations of the second order has now been thoroughly developed [2]. The quasi-classical quantization rules determining the spectrum of eigenvalues of such equations are written in the form of the well-known Bohr-Sommerfeld integrals [3]. However, in analyzing the spectrum of oscillations of an inhomogeneous plasma it is insufficient in many cases to confine oneself to equations of the second order. For example, in an inhomogeneous magnetoactive plasma, even when the thermal motion of the particles is neglected, the field equations, generally speaking, reduce to a differential equation of the fourth order. Equations of the fourth order also arise in investigating the stability of the hydrodynamical flow of a viscous fluid [4].

Certain special forms of fourth-order equations were studied in [4–6]. The authors of [6] obtained a quasi-classical quantization rule for equations of the fourth order with a small parameter associated with the leading derivative. The present paper investigates the general fourth-order equation with real coefficients. Asymptotic solutions of such an equation are obtained with an accuracy to terms of the first order in the approximation of geometrical optics, and quasi-classical quantization rules are established for various concrete cases. Using the theory thus developed, a new spectrum of oscillations is determined, characteristic only for an inhomogeneous plasma in a magnetic field.

1. The fourth-order equation arising from the investigation of small oscillations of an inhomogeneous plasma in an external magnetic field, dissipative processes being disregarded, may be written in its most general form in the first approximation of geometrical optics as follows:

$$y^{\mathrm{IV}} + 2p(\omega, x)\, y'' + 2\varepsilon(\omega, x)\, y' + q(\omega, x)\, y = 0. \quad (1.1)$$

Here $p(\omega, x)$, and $q(\omega, x)$ are slowly varying real functions of the x coordinate, so that $\delta \sim p'p^{-3/2} \sim q'q^{-5/4} \ll 1$ over the whole region of variation of x, the real function $\varepsilon(\omega, x)$ is small compared with $p(\omega, x)$ and $q(\omega, x)$ and is of the first order of smallness in the parameter δ, and, finally, ω is an eigenvalue. The functions $p(\omega, x)$, $q(\omega, x)$ and $\varepsilon(\omega, x)$ are real for real x (or almost real).

We shall seek solutions of equation (1.1) with an accuracy to terms of the first order in the parameter δ. We shall write the required functions in the form

$$y = C \exp\left\{i \int^{x} k(\omega, x)\, dx\right\}. \quad (1.2)$$

Then in the zeroth approximation of geometrical optics (i.e., with respect to the parameter δ) we obtain for the function $k(\omega, x)$, called the wave vector,

$$k_{1,2}^{2} = p \pm \sqrt{p^{2} - q}. \quad (1.3)$$

We find the following correction in the first approximation

$$\delta k = \frac{i}{2}\left\{[\ln k(p^{2} - q)]' + \frac{p' - \varepsilon}{k^{2} - p}\right\}. \quad (1.4)$$

It is clear from this expression that geometrical optics is violated close to the points

$$k(\omega, x) = 0, \qquad p^{2}(\omega, x) = q(\omega, x). \quad (1.5)$$

The first of these singular points has already been met in the theory of differential equations of the second order and is called a turning point. However, singular points of the second type are characteristic only for equations of the fourth order. In what follows such singular points will be called branch points. It will be shown that the presence of branch points may in certain cases lead to the appearance of a small imaginary part in the eigenvalues, which corresponds to weak (in the first approximation of geometrical optics) damping, or increase of the oscillations described by equation (1.1). In addition to this, as a result of the wave vectors k_1 and k_2 coinciding at the branch points a transformation of the different eigensolutions corresponding to the wave vectors k_1 and k_2 occurs, and this in its turn leads to the linking of the wave vectors of the different waves in the quantization rules of the zeroth approximation of geometrical optics.

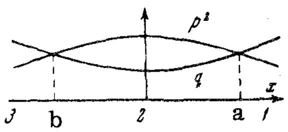

Fig. 1

result of the wave vectors k_1 and k_2 occurs, and this in its turn leads to the linking of the wave vectors of the different waves in the quantization rules of the zeroth approximation of geometrical optics.

We shall consider the case when the branch points lie on the real axis with no other singular points between them where geometrical optics is violated (Fig. 1). We shall also assume that $p(\omega, x) > 0$. In region 2 (region of transparency) remote from the branch points we may write the general solution of equation (1.1) in the form

$$y^{(2)} = \frac{C_1}{\sqrt[4]{k_1^{2}(p^{2} - q)}} \exp\left(i \int^{x} k_1\, dx - \frac{1}{2} \int^{x} \frac{p' - \varepsilon}{\sqrt{p^{2} - q}}\, dx\right) + \quad (1.6)$$

$$\frac{C_2}{\sqrt[4]{k_1^2(p^2-q)}}\exp\left(-i\int^x k_1dx-\frac{1}{2}\int^x\frac{p'-\varepsilon}{\sqrt{p^2-q}}dx\right)+$$
$$+\frac{C_3}{\sqrt[4]{k_2^2(p^2-q)}}\exp\left(i\int^x k_2dx+\frac{1}{2}\int^x\frac{p'-\varepsilon}{\sqrt{p^2-q}}dx\right)+ \qquad (1.6)$$
$$+\frac{C_4}{\sqrt[4]{k_2^2(p^2-q)}}\exp\left(-i\int^x k_2dx+\frac{1}{2}\int^x\frac{p'-\varepsilon}{\sqrt{p^2-q}}dx\right) \qquad \text{(cont'd)}$$

with an accuracy to terms of the first order in the geometrical optics approximation.

In regions 1 and 3 (nontransparent region), the increasing solutions must be rejected, since solutions must be finite for $x = \pm\infty$. We then have

$$y^{(1)}=\frac{C_1'}{\sqrt[4]{\bar{k}_1^2(q-p^2)}}\exp\left(i\int_a^x\bar{k}_1dx+\frac{i}{2}\int_a^x\frac{p'-\varepsilon}{\sqrt{q-p^2}}dx\right)+$$
$$+\frac{C_4'}{\sqrt[4]{\bar{k}_2^2(q-p^2)}}\exp\left(-i\int_a^x\bar{k}_2dx-\frac{i}{2}\int_a^x\frac{p'-\varepsilon}{\sqrt{q-p^2}}dx\right),$$
$$y^{(3)}=\frac{C_2'}{\sqrt[4]{\bar{k}_1^2(q-p^2)}}\exp\left(+i\int_x^b\bar{k}_1dx+\frac{i}{2}\int_b^x\frac{p'-\varepsilon}{\sqrt{q-p^2}}dx\right)+$$
$$+\frac{C_3'}{\sqrt[4]{\bar{k}_2^2(q-p^2)}}\exp\left(-i\int_x^b\bar{k}_2dx-\frac{i}{2}\int_b^x\frac{p'-\varepsilon}{\sqrt{q-p^2}}dx\right),$$
$$\bar{k}_{1,2}^2=p\pm i\sqrt{q-p^2}\,. \qquad (1.7)$$

In order to match solutions (1.6) and (1.7), each valid on different sides of the branch points a and b, we make a formal continuation of these solutions to the complex plane of x and go around the points a and b following a contour such that the conditions of applicability of the geometrical optics approximation are fulfilled. Below we give a proof of the validity of such a method of matching the solutions. Bypassing the branch points by semi-circumference of large radius, situated in the upper and lower half planes, we obtain the relation between the coefficients C_i, C_i' and C_i° ($i = 1, 2, 3, 4$). The coefficients C_i correspond to the solution of (1.6) close to the point a and the coefficients C_i° to that close to the point b (see, for example, [7], §47). We find

$$C_1'e^{-1/4 i\pi}=C_3,\quad C_4'e^{-1/4 i\pi}=C_2,$$
$$C_1'e^{1/4 i\pi}=C_1,\quad C_4'e^{1/4 i\pi}=C_4$$

on going around the point a, and

$$C_2'e^{1/4 i\pi}=C_2^\circ,\quad C_3'e^{1/4 i\pi}=C_3^\circ,$$
$$C_2'e^{-1/4 i\pi}=C_4^\circ,\quad C_3'e^{-1/4 i\pi}=C_1^\circ$$

on going around the point b.

Here each of the two linear independent solutions in the nontransparent region separates into only two independent solutions in the transparent region on going around the branch point. As a result we obtain two asymptotic solutions of equation (1.1) in the transparent region, and, on being identified, these lead to a system of four linear equations for the coefficients C_i'. From the condition of solvability of this system we obtain the required quantization rules which determine the spectrum of eigenvalues for equation (1.1) when two branch points are present

$$\int_b^a(k_1-k_2)\,dx\pm i\int_b^a\frac{p'-\varepsilon}{\sqrt{p^2-q}}\,dx=2\pi\left(n+\frac{1}{2}\right), \qquad (1.8)$$

where n is an integer. In order that nontrivial solutions of equation (1.1) should exist, it suffices that one of the relations (1.8) be fulfilled. This means that the given relations determine two sets of eigenvalues ω, corresponding to the two different systems of eigensolutions of equation (1.1). It follows from relation (1.8) that if $p' \neq \varepsilon$ over the whole region of transparency, then the spectrum of eigenvalues ω possesses a small imaginary part (of order δ), corresponding to solutions which are damped or increasing with time.

2. We shall now give a stricter mathematical justification of the method of matching solutions given above. The method explained above is based on the assumption that it is always possible to choose a contour on which the condition for geometrical optics to be applicable is nowhere violated on going around the singular points in the complex x plane. We shall show that this is in fact the case. We shall suppose without limiting the generality of the proof that $\varepsilon(\omega, x) = 0$. Then in the neighborhood of a branch point equation (1.1) may be written approximately in the form

$$y^{\mathrm{IV}}+(2p_0-\beta x)\,y''+(p_0^2+\alpha x)\,y=0 \qquad (2.1)$$

where $\beta = -2p_0'$ and $\alpha = q_0'$. To be specific, we shall assume that $\alpha > 0$ and $\beta > 0$. The exact solution of equation (2.1) is found by Laplace's method (see, for example, [7])

$$y(x)=\int_C\frac{Z(t)}{Q(t)}\,dt \qquad (2.2)$$

where

$$Z(t)=\exp\left\{xt+\int^t\frac{P(t')}{Q(t')}\,dt'\right\},$$
$$P(t)=(p_0+t^2)^2,\quad Q(t)=\alpha-\beta t^2. \qquad (2.3)$$

The integration in (2.2) is carried out in the complex plane over contours C on which the function Z(t) returns to its initial value after describing the entire line C. In the case under consideration there are four such linear independent contours, which correspond to four linear independent solutions of equation (2.1). The integrals (2.2) may be calculated by the method of steepest descent for large values of x. Denoting the saddle points by $t_i(x)$ ($i = 1, 2, 3, 4$), we may write the asymptotic solutions in the form

$$y(x) = \sum_i \frac{1}{x - 3t_i^2} \exp\left(\int^x t_i(x)\,dx\right) \times$$
$$\times \int_C dt \exp\left\{\frac{1}{2}\left(\frac{P}{Q}\right)'_{t_i} (t - t_i)^2\right\}. \qquad (2.4)$$

For a start we shall consider the case $p'(\omega, x) = 0$. For this case the integration contours in the t plane are illustrated in Fig. 2: (1) and (2) for $x > 0$, and (3) and (4) for $x < 0$. In order for the integrals (2.2) to be finite, the contours are chosen in such a manner that they go to infinity in the hatched sectors, in which $Z(t) \to 0$ for $t \to \infty$. The saddle points for $x > 0$ (nontransparent region) are marked by stars in Fig. 2, and for $x < 0$ (transparent region) by squares, while the integration contours are drawn through these in the direction of steepest descent. The solutions corresponding to the contours passing through the points t_1 and t_4 for $x > 0$, diverge at infinity. Consequently, such contours are not shown in Fig. 2. Finally, we note that on approaching the branch point

$$\pm k_2 \to t_{1,3}, \quad \pm k_1 \to t_{2,4} \quad \text{for } x > 0$$
$$\pm k_2 \to t_{1,3}', \quad \pm k_1 \to t_{2,4}' \quad \text{for } x < 0.$$

The direct calculation of (2.4) over the contours shown in Fig. 2 leads to solutions which, when these relations are taken into account, pass into the corresponding quasi-classical solutions (1.6) and (1.7),

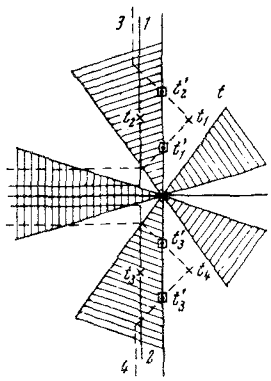

Fig. 2

matched by the method set out above. It remains for us to prove that the solutions obtained by integration over contours (1) and (2) for $x > 0$, pass respectively into the solutions obtained by means of integration over contours (3) and (4) for $x < 0$. Specifying a finite solution for $x > 0$ (which is equivalent to giving a contour of integration) unambiguously determines the solution for $x < 0$, if the ends of the contours determining these solutions go to infinity in the complex t plane in one and the same sectors. This is so for contours (1), (3), (2), (4), which we have chosen. In order to prove that the solutions are finite, it was shown that the real part of the exponent in the integrand of (2.2) on the segments of the lines composing these contours has a maximum at the saddle points and falls off exponentially as it departs from them in the direction of steepest descent. This shows that the asymptotic expression (2.4) is valid and, at the same time, confirms the method explained above of matching the quasi-classical solutions of equation (1.1). Thus (1.6), (1.7) and (2.2) determine the solutions of equation (1.1) (for $p' = \varepsilon = 0$) over the whole region of variation of x.

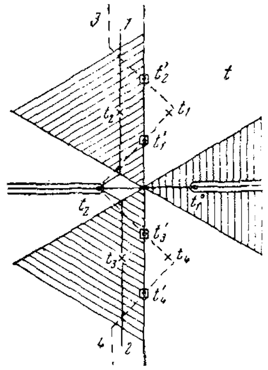

Fig. 3

In the case $p' \neq 0$ the principle remains the same, but the contours have a somewhat different appearance, which is associated with the presence of a branch point in the integrand of function (2.2) for $\beta \neq 0$, $t_{1,2}^\circ = \pm(\alpha / \beta)^{1/2}$ with an integrable singularity at the point t_2°. In order for the integrand of the function to be single-valued in the t plane branch cuts are made along the real axis (see Fig. 3). The continuous and the broken curves indicate the integration contours we have chosen for $x > 0$ and $x < 0$, respectively, which do not now go to infinity parallel to the real axis, but terminate at the point t_2°, where $Z(t_2^\circ) = 0$.

3. In the preceding sections we obtained the quantization rules which determine the spectrum of eigenvalues of equation (1.1) in the case when there are only two branch points in the region of variation of z and the transparent region is situated between them. We shall now give the quantization rules for other cases without proof. Such quantization rules are easily obtained by the method described above of matching the quasi-classical solutions (1.6) and (1.7).

If the region of variation of x has only one branch point a, and nondissipative boundary conditions $y(b) = y'(b) = 0$ are given on the other boundary of the transparent region, then the quantization rule has the form

$$\int_b^a (k_1 - k_2)\,dx = \pi n, \qquad (3.1)$$

where n is an integer much larger than unity. In relation (3.1) small terms of the first order in the approximation of geometrical optics are discarded.

As opposed to (1.8), they turn out to be real in the case under consideration and lead to insignificant corrections.

When there is a turning point $q(\xi) = 0$ (k_2^2 changes sign at this point) between the branch point a and the boundary of the region of variation of x, for which the condition $y(b) = 0$ is given, the quantization rule determining the spectrum of eigenvalues of (1.1) is written in the form

$$\int_b^a k_1\,dx - \int_\xi^a k_2\,dx = \pi\left(n - \frac{1}{4}\right). \qquad (3.2)$$

This relation has been written down with an accuracy to terms of the first order, inclusive.

Finally, if the region of transparency is bounded only by turning points or by a boundary for which nondissipative boundary conditions are given, then the quantization rules for each of the wave-vectors k_1 and k_2 have the same form as in the case of the second-order equation [3].

4. We shall now apply the quantization rules obtained above to real oscillations of a spatially inhomogeneous plasma. Naturally, the case when branch points exist in the region of variation of x (the region occupied by the plasma) is of greatest interest. We noted above that coupling of the various modes of oscillation occurs at the branch points, which should lead to a qualitative alteration of the oscillation spectrum of an inhomogeneous plasma compared to a homogeneous one.

We shall consider the potential oscillations of a magnetoactive Maxwellian plasma in the frequency region $\Omega_i \ll \omega \ll \Omega_e$ (Ω is the Larmor frequency of the particles, the indices e and i refer to the electrons and ions, respectively), when the electrons are strongly affected by magnetic forces but the ions are free. We let the magnetic field be directed along the z axis and assume that $\omega \gg k_z v_e$, kv_i, and $k_\perp = \sqrt{k_y^2 + k_x^2} \gg k_z$ (**k** is the wave-vector, and $v = \sqrt{T/m}$ is the thermal velocity of the particles). Finally, the plasma is taken to be nonisothermal with the ions hotter than the electrons, i.e., $T_i \gg T_e$. Under these conditions the dielectric constant, which describes the spectrum of potential oscillations of a spatially homogeneous plasma, is given in [8]

$$\varepsilon^l = 1 + \frac{k_\perp^2}{k^2}\frac{\omega_e^2}{\Omega_e^2} - \frac{k_z^2}{k^2}\frac{\omega_e^2}{\omega^2} - \frac{\omega_i^2}{\omega^2}\left(1 + 3\frac{k^2 v_i^2}{\omega^2}\right) \qquad (4.1)$$

where ω_e and ω_i are the plasma frequencies of the electrons and ions. The plasma oscillation spectrum is determined by the zeros of this expression and has the form

$$\omega^2 = \frac{k_z^2\omega_e^2 + k_\perp^2\omega_i^2}{k_\perp^2(1 + \omega_e^2/\Omega_e^2)} + 3\frac{\omega_i^2 k_\perp^4 v_i^2}{k_z^2\omega_e^2 + k_\perp^2\omega_i^2}. \qquad (4.2)$$

We shall now show that under the conditions being considered a new spectrum distinct from (4.2) appears in a spatially inhomogeneous plasma when branch points are present. We shall restrict ourselves to a weakly inhomogeneous plasma with irregularities along the x axis. Starting from the equations of two-fluid magnetohydrodynamics with weak ionic pressure [8], we obtain the following differential equation for the potential of the oscillation field in the frequency range under consideration

$$3\frac{\omega_i^2 v_i^2}{\omega^4}\Phi^{IV} + 6\left(\frac{\omega_i^2 v_i^2}{\omega^4}\right)'\Phi''' + \left[1 + \frac{\omega_e^2}{\Omega_e^2} - \frac{\omega_i^2}{\omega^2} - \right.$$

$$\left. - 6(k_y^2 + k_z^2)\frac{\omega_i^2 v_i^2}{\omega^4}\right]\Phi'' + \left[1 + \frac{\omega_e^2}{\Omega_e^2} - \frac{\omega_i^2}{\omega^2} - \right.$$

$$\left. - 6(k_y^2 + k_z^2)\frac{\omega_i^2 v_i^2}{\omega^4}\right]'\Phi' - \left[k_z^2\left(1 - \frac{\omega_e^2}{\omega^2}\right) - \right.$$

$$- k_y\left(\frac{\omega_e^2}{\omega\Omega_e}\right) + k_y^2\left(1 + \frac{\omega_e^2}{\Omega_e^2} - \frac{\omega_i^2}{\omega^2}\right) -$$

$$\left. - 3\frac{\omega_i^2 v_i^2}{\omega^4}(k_y^2 + k_z^2)^2\right]\Phi = 0. \qquad (4.3)$$

This equation easily reduces to an equation of the form of (1.1) with a change of variables. Here it turns out that $p' = \varepsilon$. Thus the quantization rule (1.8) for this equation is written in the form

$$\int_b^a (k_1 - k_2) = 2\pi n, \qquad (4.4)$$

where

$$k_{1,2}^2 = -k_y^2 + \frac{\omega^4}{6\omega_i^2 v_i^2}\left\{1 + \frac{\omega_e^2}{\Omega_e^2} - \frac{\omega_i^2}{\omega^2} \pm \right.$$

$$\left. \pm\left[\left(1 + \frac{\omega_e^2}{\Omega_e^2} - \frac{\omega_i^2}{\omega^2}\right)^2 - 12\frac{k_z^2 v_i^2\omega_e^2\omega_i^2}{\omega^6}\right]^{1/2}\right\}. \qquad (4.5)$$

In writing these relations, terms of the first order in the parameter of geometrical optics were neglected, and the limitations on the oscillation frequency indicated above were taken into account. From expression (4.5) it is clear that in an inhomogeneous plasma with a pressure which decreases towards the plasma boundary, or with an increasing magnetic field, two branch points may be present with a transparent region situated between them. Using a formula for the average value of an integral we can show from (4.4) and (4.5) that for an inhomogeneous plasma the characteristic frequencies of oscillations enclosed between branch points are determined with a good degree of accuracy by the zeros of the expression under the square root in (4.5), under the conditions considered. In addition to this, taking into account the fact that $p > 0$ for the determination of positive solutions $\omega^2 > 0$ (it is only here that branch points exist), we obtain the approximate equation

$$\left(1 + \frac{\omega_e^2}{\Omega_e^2}\right)\omega^4 - \omega_i^2\omega^2 -$$

$$- \sqrt{12\omega^2 k_z^2 v_i^2\omega_e^2\omega_i^2} = 0. \qquad (4.6)$$

Hence it is clear that as the wave number k_z increases the frequencies of the oscillations enclosed within the branch points of an inhomogeneous plasma vary within the limits

$$\frac{\omega_i^2}{1+\omega_e^2/\Omega_e^2} \leqslant \omega^2 \leqslant \left[\frac{12k_z^2 v_i^2 \omega_e^2 \omega_i^2}{(1+\omega_e^2/\Omega_e^2)^2}\right]^{1/3} \leqslant$$

$$\leqslant 12k_z^2 v_e^2 \frac{T_i}{T_e}. \qquad (4.7)$$

For $k_z \to 0$ the spectrum of inhomogeneous plasma oscillations under investigation is close to the oscillation spectrum of a homogeneous plasma (4.2). However, as k increases it departs more and more from the spectrum (4.2), while in the shortwave limit when

$$k_z^2 v_i^2 \gtrsim \frac{m}{M}\,\omega_i^2,$$

it depends significantly on the ion temperature. Such oscillations are absent in a homogeneous plasma.

In conclusion, the authors express their gratitude to V. P. Silin who suggested the idea of matching the quasi-classical solutions, and also to Yu. N. Dnestrovskii and D. P. Kostomarova for discussing the paper and offering valuable criticism.

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